describe an arc cutting $X'Y'$ at $A$. With $AB$ as radius and $X$ as center describe an arc cutting $XO$ at $P$ and extending on each side of $XO$ toward $Y$ and $Y'$. With the same radius ($AB$) and $P$ as center draw an arc through $X$ cutting the the arc just drawn in the points $M$ and $N$. Construct in similar manner the arc $M'N'$ through $X'$. With $NN'$ as radius and $N$ and $N'$ as centers describe arcs toward $Y'$ cutting each other, obviously, in some point $R$ on $OY'$ (produced, if necessary). With $R$ as center and $RN$ as radius draw the arc $NN'$. In similar manner draw the arc $MM'$. The curve $XMY'M'X'N'YNX$ is the required approximation.

*Note.*—The above construction obviously can not be a true ellipse, except for $a = b$, since the curvature on the arc $NYN'$ is constant while the curvature of the ellipse is variable $(= a^4b^4/(b^4x^2 + a^4y^2)^{3/2}$. This suggests that the construction is a close approximation only in case the difference $a - b$ is small. There are, however, important applications of ellipses, such, for example, as the paths of the planets, in which the difference $a - b$ is small. In such cases Mr. Iwerson's construction might give valuable results if there were some definite criterion (possibly some function of the difference $a - b$ or of the eccentricity of the ellipse) which would readily and accurately measure the degree of approximation. Question 33 above is asked in the hope that some of our readers may be able to furnish such a criterion or suggest useful applications.—U. G. M.

## DISCUSSIONS.

### Relating to the Number of Terms between Two Given Terms of an Ordered Polynomial.

By O. E. Glenn, University of Pennsylvania.

A complete homogeneous polynomial $P$ of order $m$ in $n$ letters $x_1, x_2, \cdots, x_n$ may be said to have its terms arranged in normal order when the exponents in any two terms of $P$, as

$$t = C_{k_1k_2\cdots k_n}x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n},\ t' = C_{l_1l_2\cdots l_n}x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n},$$

where $t$ comes before $t'$, satisfy the condition that the first non-vanishing difference of the set

$$k_n - l_n,\ k_{n-1} - l_{n-1},\ \cdots,\ k_p - l_p,\ \cdots,\ (k_1 - l_1),$$

is negative. We wish to prove that the number of terms of $P$ between $t$ and $t'$, including $t'$ but not including $t$, is given by the formula

$$\text{(2)}\qquad N\binom{k_1,\ \cdots,\ k_n}{l_1,\ \cdots,\ l_n} = \left\{\begin{array}{l} \binom{l_1}{0} + \binom{l_1+1}{0} + \cdots + \binom{k_1-1}{0} \\ + \binom{l_1+l_2+1}{1} + \binom{l_1+l_2+2}{1} + \cdots + \binom{k_1+k_2}{1} \\ + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ + \binom{l_1+\cdots+l_{n-2}+n-3}{n-3} + \binom{l_1+\cdots+l_{n-2}+n-2}{n-3} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-2}+n-4}{n-3} \\ + \binom{l_1+\cdots+l_{n-1}+n-2}{n-2} + \binom{l_1+\cdots+l_{n-1}+n-1}{n-2} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-1}+n-3}{n-2} \end{array}\right\},$$

where the numbers $\binom{a}{b}$ are ordinary binomial numbers, and are defined to be zero when $b$ is negative.

All terms in this formula have plus signs provided

$$l_1 + \cdots + l_s < k_1 + \cdots + k_s (s = 1, \cdots, n - 1).$$

If, for any of these values of $s$, $l_1 + \cdots + l_s = k_1 + \cdots + k_s$, the line number $s$, viz.,

$$\binom{l_1 + \cdots + l_s + s - 1}{s - 1} + \binom{l_1 + \cdots + l_s + s}{s - 1} + \cdots + \binom{k_1 + \cdots + k_s + s - 2}{s - 1}$$

disappears from the formula. But if $l_1 + \cdots + l_s > k_1 + \cdots + k_s$ for a value of $s$, then line number $s$ is to be replaced by, or is to be interpreted as signifying

$$-\binom{k_1 + \cdots + k_s + s - 1}{s - 1} - \binom{k_1 + \cdots + k_s + s}{s - 1} - \cdots - \binom{l_1 + \cdots + l_s + s - 2}{s - 1}.$$

We have $k_1 + k_2 + \cdots + k_n = l_1 + l_2 + \cdots + l_n = m$, and $P$ is ordered according to ascending powers of $x_n$, the coefficients of the various powers of $x_n$ being polynomials in $n - 1$ letters $x_1, \cdots, x_{n-1}$, *i. e.*, in a common terminology, are $(n - 1)$-ary forms. Beginning to enumerate terms from the lower (right-hand) end of $P$ we observe that $N\begin{pmatrix} k_1, & k_2, & \cdots, & k_n \\ 0, & 0, & \cdots, & m \end{pmatrix}$ equals 1 plus the number of terms in an $(n - 1)$-ary linear form, plus the number of terms in an $(n - 1)$-ary quadratic form, plus the number in a cubic, plus and so on to and including the number of terms in an $(n - 1)$-ary $(m - k_n - 1)$-ic form; plus 1 plus the number of terms in an $(n - 2)$-ary linear form, plus the number in an $(n - 2)$-ary quadratic and so on to and including the number in an $(n - 2)$-ary $(m - k_n - k_{n-1} - 1)$-ic form, and so forth. This double entry summation extends to include the (last) row 1 plus the number of terms in a linear unary form, plus the number in a quadratic unary form, plus and so on to and including the number in a unary $(m - k_n - k_{n-1} - \cdots - k_2 - 1)$-ic form. Hence

$$(3) \quad N\begin{pmatrix} k_1, & \cdots, & k_n \\ 0, & \cdots, & m \end{pmatrix} = \left\{ \begin{array}{l} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{k_1 - 1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{k_1 + k_2}{1} \\ + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + \binom{n-3}{n-3} + \binom{n-2}{n-3} + \cdots + \binom{k_1 + \cdots + k_{n-2} + n - 4}{n - 3} \\ + \binom{n-2}{n-2} + \binom{n-1}{n-2} + \cdots + \binom{k_1 + \cdots + k_{n-1} + n - 3}{n - 2} \end{array} \right\}.$$

This serving as a general formula we have immediately

$$N\begin{pmatrix} k_1, & \cdots, & k_n \\ 0, & \cdots, & m \end{pmatrix} - N\begin{pmatrix} l_1, & \cdots, & l_n \\ 0, & \cdots, & m \end{pmatrix} = N\begin{pmatrix} k_1, & \cdots, & k_n \\ l_1, & \cdots, & l_n \end{pmatrix},$$

equal to formula (2) above.

For illustration, a complete quaternary cubic in normal order is the following:

$$\begin{aligned} & x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 \\ & + x_1x_3^2 + x_2x_3^2 + x_3^3 + x_1^2x_4 + x_1x_2x_4 + x_2^2x_4 + x_1x_3x_4 \\ & + x_2x_3x_4 + x_3^2x_4 + x_1x_4^2 + x_2x_4^2 + x_3x_4^2 + x_4^3. \end{aligned}$$

The number of terms between $x_1^2x_3$ and $x_3x_4^2$ is

$$N\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} \\ + \binom{1}{1} + \binom{2}{1} \\ + \binom{3}{2} + \binom{4}{2} \end{array} \right\} = 14.$$

The total number of terms in a complete ternary form of order $m$ is

$$1 + N\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & m \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{m-1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{m}{1} \end{array} \right\} + 1 = \tfrac{1}{2}(m+1)(m+2).$$

Exercise. Show that the number of terms between $x_1^3x_2^2x_3^2x_4^{m-7}$ and $x_1^2x_2x_3^2x_4^{m-5}$ in a complete ordered quaternary polynomial of order $m$ ($m > 6$) is 59.

Exercise. Show that

$$N\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \end{pmatrix} = 31.$$

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# NOTES AND NEWS.

Send communications to D. A. Rothrock, Indiana University.

Dr. Otto Dunkel, formerly instructor at the University of Missouri, has been appointed assistant professor of mathematics at Washington University, St. Louis, Mo.

"Differential equations and implicit functions in infinitely many variables" is the title of a paper by Dr. W. L. Hart of Harvard University, abstract of which appeared in the *Proceedings* of the National Academy of Sciences for June, 1916.

Mr. Edward B. Escott is now connected with the auditing department of the Kansas City Life Insurance Company, Kansas City, Mo.

Mr. H. C. Clevenger, of Urbana, Ill., has accepted an instructorship in mathematics at the University of Minnesota.

Dr. R. L. Börger, associate in mathematics at the University of Illinois, goes to Ohio University at Athens as professor of mathematics.